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ON A LOCAL PROPERTY OF PRODUCTS OF LAŠNEV SPACES

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Let p denote a free ultrafilter on the natural numbers N. It is shown that $N \cup \{p\}$ cannot be embedded in any countable product of Lašnev spaces.

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product Lašnev space ultrafilter on N

1. Introduction

A *Lašnev space* is a space which is the image of a metric space under a closed mapping. It is well known that a product of Lašnev spaces behaves quite badly with respect to local properties. Even a product of two Lašnev spaces does not preserve Fréchet property, sequentiality and countable tightness. (See [2], [6], [1; 2.3].)

An interesting general problem is to determine which neighborhood filters can be embedded in a finite or countable product of Lašnev spaces.

Let p be a free ultrafilter on the natural numbers N. In this note it is shown that $N \cup \{p\}$ cannot be embedded in any countable product of Lašnev spaces. This answers the problem of T. Nogura [5].

K. Nagami defined the class of *free L-spaces* and in that class he proved fundamental theorems in dimension theory [4]. The class is an intermediate class between that of Lašnev spaces and that of M_1-spaces. He posed the following problem: Does there exist a free L-space which cannot be embedded in the product of countably many Lašnev spaces?

Since $N \cup \{p\}$ is a free L-space, our result also answers Nagami's question.

Before starting the proof of our theorem, note that we need only show that $N \cup \{p\}$ cannot be embedded in any finite product of Lašnev spaces because of the following theorem:

Theorem 1 (Nogura [5]). *Let $\{X_n : n \in N\}$ be a family of spaces and p be a free ultrafilter on N. If $N \cup \{p\}$ can be embedded in $\prod_{n=1}^{\infty} X_n$, then there exists $n(0) \in N$ such that $N \cup \{p\}$ can be embedded in $\prod_{n=1}^{n(0)} X_n$.*

2. Embedding into a diagonal

A space with only one non-isolated point is denoted by $D \cup \{q\}$, where q is the non-isolated point.

In this section, we shall show that if $D \cup \{q\}$ can be embedded in a finite product of Lašnev spaces under certain conditions, then we can get a diagonal embedding into another product of Lašnev spaces.

Let $\pi_i : \prod_{i=1}^{n} X_i \to X_i$ be projections.

Theorem 2. *Suppose that $\phi : D \cup \{q\} \to \prod_{i=1}^{n} X_i$ is an embedding into a finite product $\prod_{i=1}^{n} X_i$ of Lašnev spaces such that $\pi_i\phi(d) \neq \pi_i\phi(q)$ for each $d \in D$ and $i = 1, \dots, n$. Then there exists a product $\prod_{i=1}^{n} (D \cup \{q_i\})$ of Lašnev spaces with only one non-isolated point and of the same cardinality such that $D \cup \{q\}$ is homeomorphic to the diagonal*

$$\Delta = \{(d, \dots, d) : d \in D\} \cup \{(q_1, \dots, q_n)\}.$$

To prove this, we need the following lemmas.

Let X be a space and $q \in X$. We denote by X_q the space with the same underlying set as X, for which each point of $X \backslash \{q\}$ is isolated and the neighborhood base at the point q in X_q is the same as in X.

Lemma 1 [5]. *Let X be a Lašnev space. Then X_q is Lašnev for each $q \in X$.*

Lemma 2. *Let $\prod_{i=1}^{n} (A_i \cup \{a_i\})$ be a product of Lašnev spaces with only one non-isolated point and $M \subset \prod_{i=1}^{n} A_i$. Then there exists another product $\prod_{i=1}^{n} (D \cup \{q_i\})$ of Lašnev spaces of the same cardinality as M, such that $M \cup \{(a_1, \dots, a_n)\}$ is homeomorphic to $M' \cup \{(q_1, \dots, q_n)\} \subset \prod_{i=1}^{n} (D \cup \{q_i\})$ and $|\pi_i^{-1}(x) \cap (M' \cup \{(q_1, \dots, q_n)\})| = 1$ for each $x \in D \cup \{q_i\}$, $i = 1, \dots, n$.*

Proof. We shall only construct $D \cup \{q_1\}$. Repeating this construction about each factor, we can get the desired result.

For each $a \in A_1$, let D_a be a set of the same cardinality as $\pi_1^{-1}(a) \cap M$ and $\phi_a : \pi_1^{-1}(a) \cap M \to D_a$ be a bijection. Let $D = \bigcup\{D_a : a \in A_1\}$ be a disjoint union and q_1 be a point which is not in D. Note that D is of the same cardinality as M. Topologize $D \cup \{q_1\}$ as follows. Each point of D is isolated. For the neighborhood base at q_1, we take $\{U^* : U$ is a neighborhood of a_1 in $A_1 \cup \{a_1\}\}$, where

$$U^* = \{q_1\} \cup \bigcup\{D_a : a \in U \cap A_1\}.$$

We claim that $D\cup\{q_1\}$ is a Lašnev space. Since $A_1\cup\{a_1\}$ is a Lašnev space, there exists a closed continuous mapping $f: X\to A_1\cup\{a_1\}$ of a metric space X onto $A_1\cup\{a_1\}$. Choose one point x_a from each $f^{-1}(a)$ for $a\in A_1$. Since $f^{-1}(a_1)\cup\{x_a: a\in A_1\}$ is closed in X, we may assume that X takes the form of $X=F\cup A_1$, where $f(F)=a_1$ and $f(a)=a$ for each $a\in A_1$.

Let $Y=F\cup D$ be a disjoint union and let $g: Y\to D\cup\{q_1\}$ be a map such that $g(F)=q_1$ and $g(d)=d$ for each $d\in D$.

Let $\{\mathcal{U}_n: n\in N\}$ be a σ-discrete open base for X. Put $\mathcal{U}'_n=\{U': U\in\mathcal{U}_n\}$, where

$$U'=(U\cap F)\cup\bigcup\{D_a: a\in U\backslash F=U\cap A_1\}.$$

Since $X\backslash F$ is an open subset of a metric space X, we can write $X\backslash F=\bigcup\{F_n: n\in N\}$ where each F_n is closed in X. Put

$$\mathcal{V}_n=\{d: d\in D_a \text{ for some } a\in F_n\}.$$

Now it is easy to verify that $\{\mathcal{U}'_n: n\in N\}\cup\{\mathcal{V}_n: n\in N\}$ forms a σ-discrete base for a metrizable topology of Y and under the topology the map g becomes closed and continuous. Hence $D\cup\{q_1\}$ is a Lašnev space.

Now put

$$M'=\{(\phi_{\pi_1(x)}(x), \pi_2(x), \ldots, \pi_n(x)): x\in M\}$$
$$\subset (D\cup\{q_1\})\times\prod_{i=2}^{n}(A_i\cup\{a_i\}).$$

Define a map

$$\psi: M\cup\{(a_1,\ldots,a_n)\}\to M'\cup\{(q_1,a_2,\ldots,a_n)\}$$

letting $\psi((a_1,\ldots,a_n))=(q_1,a_2,\ldots,a_n)$ and $\psi(x)=(\phi_{\pi_1(x)}(x),\pi_2(x),\ldots,\pi_n(x))$ for each $x\in M$. Then the map ψ is a homeomorphism and

$$|\pi_1^{-1}(y)\cap(M'\cup\{(q_1,a_2,\ldots,a_n)\})|=1 \quad \text{for each } y\in D\cup\{q_1\}.$$

This completes the proof.

Proof of Theorem 2. Suppose $\phi: D\cup\{q\}\to\prod_{i=1}^n X_i$ is an embedding into a product of Lašnev spaces such that $\pi_i\phi(d)\neq\pi_i\phi(q)$ for each $d\in D$ and $i=1,\ldots,n$. By Lemma 1, we may assume that each X_i is a space with only one non-isolated point. Put $\phi(D)=M$ and $\phi(q)=(a_1,\ldots,a_n)$. Then the condition of Lemma 2 is fulfilled. Thus we get an embedding $\phi': D\cup\{q\}\to\prod_{i=1}^n(D\cup\{q_i\})$ such that $|\pi_i^{-1}(x)\cap\phi'(D\cup\{q\})|=1$ for each $x\in D\cup\{q_i\}$, $i=1,\ldots,n$.

Note that each $\pi_i\phi': D\cup\{q\}\to D\cup\{q_i\}$ is a bijection. Let q'_i be a point which is not in D and define a function $\psi_i: D\cup\{q_i\}\to D\cup\{q'_i\}$ letting $\psi_i(q_i)=q'_i$ and $\psi_i(x)=(\pi_i\phi')^{-1}(x)$ for each $x\in D$. Define a topology on $D\cup\{q'_i\}$ to make ψ_i be a homeomorphism. Then the map $\psi: D\cup\{q\}\to\prod_{i=1}^n(D\cup\{q'_i\})$ defined by $\pi_i\psi=\psi_i\pi_i\phi'$ is the desired diagonal embedding.

3. Impossibility of the embedding

Theorem 3. *Let p be a free ultrafilter on N. Then $N \cup \{p\}$ cannot be embedded in any finite product of Lašnev spaces.*

Let βN denote the Stone–Čech compactification of N.

It is convenient to discuss our problem in βN. For each $M \subset N$, let $M^* = \mathrm{Cl}_{\beta N} M \setminus N$. Let F be a closed subset of N^*. We introduce a topology of $X = N \cup \{F\}$ as follows: Each point of N is isolated and a neighborhood filter of $\{F\}$ in X is $\{(N \cap U) \cup \{F\}: U \in \mathcal{U}_F\}$, where $\mathcal{U}_F$ is the neighborhood filter of F in βN.

A countable space with only one non-isolated point is denoted by $N \cup \{q\}$. Here q is the non-isolated point and its filter of neighborhood restricted to N is denoted by $\mathcal{F}_q$. We define the *representation* of q in N to be the set

$$F_q = \bigcap \{\mathrm{Cl}_{\beta N} F: F \in \mathcal{F}_q\} \subset N^*.$$

Clearly $N \cup \{F_q\}$ is homeomorphic to $N \cup \{q\}$.

Lemma 3 [5]. *Let $X_i = N \cup \{F_i\}$ for each $i = 1, \ldots, n$, where each F_n is a closed subset of N^*. If there exists $M \subset N^n$ such that the neighborhood filter of $\prod_{i=1}^n \{F_i\}$ restricted to M is an ultrafilter on M, then $(\mathrm{Cl}_{(\beta N)^n} M) \cap \prod_{i=1}^n F_i$ is a singleton.*

A space X is said to be *Fréchet* if whenever $x \in \mathrm{Cl}_X A$ for some $A \subset X$, there exists a sequence $\{x_n: n \in N\} \subset A$ converging to x. A Lašnev space is known to be Fréchet.

Lemma 4 [3]. *Let F be a closed subset of N^*. Then $N \cup \{F\}$ is a Fréchet space if and only if $F = \mathrm{Cl}_{N^*}(\mathrm{Int}_{N^*} F)$.*

Lemma 5 [5]. *Let $X = N \cup \{F\}$ be a Lašnev space. Then for each $p \in F$, there exists a zero set Z_p in N^* such that $p \in Z_p \subset F$ or otherwise $p \in Z_p \subset N^* \setminus \mathrm{Int}_{N^*} F$.*

Lemma 6. (1) *Let F be a closed set and Z be a zero set in N^*. Suppose that there exists a point p such that $p \in Z$ and $p \in \mathrm{Cl}_{N^*}(F \setminus \{p\})$. Then the set $F \cap Z$ is infinite.*

(2) *Let F and H be closed sets and Z be a zero set in N^* such that $F \cap H \subset Z$. Then $\mathrm{Cl}(F \setminus Z) \cap \mathrm{Cl}(H \setminus Z) = \emptyset$.*

(3) *Let F_i be closed sets for $i = 1, \ldots, n$ and Z be a zero set in N^*. Suppose $p \in \mathrm{Cl}(F_i \setminus Z)$ for each i. Then $p \in \mathrm{Cl}(\bigcap_{i=1}^n F_i \setminus Z)$.*

Proof. (1) If Z is a clopen set it is clear. So we can assume that $Z = \bigcap \{A_n^*: n \in N\}$, $A_n \subset N$, $A_1^* \supsetneqq A_2^* \supsetneqq \cdots$, and $(A_n^* \setminus A_{n+1}^*) \cap F \neq \emptyset$ for every $n \in N$. Choose a point $x_n \in (A_n^* \setminus A_{n+1}^*) \cap F$ for each $n \in N$. Then $\mathrm{Cl}_{N^*}\{x_n: n \in N\} \setminus \{x_n: n \in N\}$ is homeomorphic to N^* and

$$\mathrm{Cl}_{N^*}\{x_n: n \in N\} \setminus \{x_n: n \in N\} \subset F \cap Z.$$

(2) Let $N^*\backslash Z = \bigcup\{A_n^* : n \in N\}$ be a disjoint union of clopen sets. Since $A_n^* \cap F \cap H = \emptyset$, there exist B_n^* and C_n^* for each $n \in N$ such that $A_n^* \cap F \subset B_n^* \subset A_n^*$, $A_n^* \cap H \subset C_n^* \subset A_n^*$ and $B_n^* \cap C_n^* = \emptyset$. Put $B = \bigcup\{B_n^* : n \in N\}$ and $C = \bigcup\{C_n^* : n \in N\}$. Then $F\backslash Z \subset B$, $H\backslash Z \subset C$, and $B \cap C = \emptyset$. It is well known that the closures of two disjoint cozero sets of N^* are again disjoint. Hence $\mathrm{Cl}_{N^*}B \cap \mathrm{Cl}_{N^*}C = \emptyset$. This proves $\mathrm{Cl}(F\backslash Z) \cap \mathrm{Cl}(H\backslash Z) = \emptyset$.

(3) For the case $n = 2$, suppose that $p \notin \mathrm{Cl}(F_1 \cap F_2\backslash Z)$. Then there exists a clopen neighborhood U of p such that $U \cap (F_1 \cap F_2\backslash Z) = \emptyset$. Put $H_1 = U \cap F_1$ and $H_2 = U \cap F_2$. Then $H_1 \cap H_2 \subset Z$. By (2), $\mathrm{Cl}(H_1\backslash Z) \cap \mathrm{Cl}(H_2\backslash Z) = \emptyset$, which contradicts the fact $p \in \mathrm{Cl}(F_1\backslash Z) \cap \mathrm{Cl}(F_2\backslash Z)$. For the general case we can prove by induction on n.

Proof of Theorem 3. We shall show that $N \cup \{p\}$ cannot be embedded in any finite product of Lašnev spaces by induction on the number of the factors. Note that $N \cup \{p\}$ cannot be embedded in any Lašnev spaces because $N \cup \{p\}$ is not Fréchet.

Now assume that $N \cup \{p\}$ cannot be embedded in any product of Lašnev spaces whose number of factors is less than n.

Suppose that $N \cup \{p\}$ can be embedded in a product $\prod_{i=1}^n X_i$ of Lašnev spaces. Let $\phi : N \cup \{p\} \to \prod_{i=1}^n X_i$ be an embedding onto a subset $M \cup \{(p_1, \ldots, p_n)\}$ of $\prod_{i=1}^n X_i$. Note that if $A \subset N$ and $p \in \mathrm{Cl}\,A$, then $A \cup \{p\}$ is homeomorphic to $N \cup \{p\}$. Since $N \cup \{p\}$ cannot be embedded in any product of Lašnev spaces whose number of factors is less than n, we have

$$(p_1, \ldots, p_n) \notin \bigcup_{i=1}^n \mathrm{Cl}(\pi_i^{-1}(\{p_i\}) \cap M).$$

Thus we may assume that $\pi_i\phi(n) \neq \pi_i\phi(p)$ for $n \in N$, $i = 1, \ldots, n$.

By Theorem 2, we can assume that the product is of the form $\prod_{i=1}^n (N \cup \{F_i\})$ where each F_i is a closed subset of N^* and $N \cup \{p\}$ is homeomorphic to the diagonal

$$\Delta = \{(n, \ldots, n) : n \in N\} \cup \{\prod_{i=1}^n \{F_i\}\}.$$

Put $\Delta_N = \{(n, \ldots, n) : n \in N\}$. Then by Lemma 3,

$$\mathrm{Cl}_{(\beta N)^n}\Delta_N \cap \prod_{i=1}^n F_i = \left\{(x, \ldots, x) \in (\beta N)^n : x \in \bigcap_{i=1}^n F_i\right\}$$

must be a singleton, which implies that $\bigcap_{i=1}^n F_i$ is a singleton.

Put $\bigcap_{i=1}^n F_i = \{p\}$. From Lemma 5, without loss of generality, we may assume that there are zero sets $\{Z_i\}_{i=1}^n$ of N^* such that $p \in Z_i \subset F_i\backslash \mathrm{Int}\,F_i$ for $i = 1, \ldots, k$ and $p \in Z_i \subset F_i$ for $i = k+1, \ldots, n$.

Put $Z = \bigcap_{i=1}^n Z_i$. From Lemma 4, $p \in \mathrm{Cl}(\mathrm{Int}\,F_i) \subset \mathrm{Cl}(F_i\backslash Z_i)$ for each $i = 1, \ldots, k$. Then by Lemma 6 (3), $p \in \mathrm{Cl}(\bigcap_{i=1}^k F_i\backslash Z)$. Thus

$$p \in Z \cap \mathrm{Cl}\left(\bigcap_{i=1}^k F_i\backslash Z\right) \subset Z \cap \mathrm{Cl}\left(\bigcap_{i=1}^k F_i\backslash\{p\}\right)$$

and $Z\cap(\bigcap_{i=1}^{k}F_i)$ must be infinite by Lemma 6(1). But $Z\cap\bigcap_{i=1}^{k}F_i\subset\bigcap_{i=1}^{n}F_i$. Contradiction.

By Theorem 1 and Theorem 3, we have,

Theorem 4. *$N\cup\{p\}$ cannot be embedded in any countable product of Lašnev spaces.*

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